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Uniformity of Deformation in Element Testing

Abstract

Uniform deformation is a basic assumption in element testing, where axial strain typically is determined from displacement measurements δ_i. In applying force or displacement, however, some rotation of the loading platen may occur such that the fundamental stress field is perturbed by bending. Thus, non-uniformity among measures of axial deformation may be present, and the response may consist of a component due to the axial force and a component due to the bending moment. To estimate this non-uniformity during an element test, at least three sensors are needed, and for equi-angular placement, it is shown that the mean of the displacement readings is equal to the displacement from the axial force; the rotation does not affect the mean value. Furthermore, the ratio of the maximum and minimum δ_i does not provide an objective evaluation of uniformity, but it is reasonable to limit the degree of rotation.

Keywords: resilient modulus, homogeneous deformation, compression testing

Introduction

An element test assumes that the material deforms in a uniform manner. For example, a specimen that is originally cylindrical in shape remains a cylinder during testing. Ideally, the kinematic boundary condition imposed by a rigid platen means that the loading platen should not rotate but remain normal to the longitudinal axis of the specimen. However, some rotation is typically allowed and when multiple displacement measurements are compared, non-uniformity between readings is inevitable. In this paper, the degree of non-uniformity due to rotation is quantified, and the relation between the degree of non-uniformity and the specimen deformation is discussed to evaluate the influence of rotation on the measured displacements (Fig. 1).

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The work is motivated by the resilient modulus (M_R) test, conducted to measure the stiffness of base and sub-grade soils of the pavement structure [1]. The resilient modulus can be thought of as Young's modulus based on the recoverable strain under repeatable axial stress. Two test protocols are commonly used: (a) Long Term Pavement Program (LTTP) P46 by the Strategic Highway Research Program (SHRP) [2], and (b) National Cooperative Highway Research Program (NCHRP) 1-28A [3]. In both protocols, repeated cycles of axial stress are applied to a specimen at a given confining pressure within a conventional triaxial cell. Each cycle is 1 s in duration, consisting of a 0.1 or 0.2 s haversine pulse followed by a 0.9 or 0.8 s rest period for coarse- or fine-grained soils.

During a resilient modulus test, force and displacement data for each sequence are collected, and the M_R is calculated from

$$M_R = \frac{\Delta\sigma_a}{\Delta\varepsilon_a} \tag{1}$$

where $\Delta\sigma_a$ = cyclic axial (deviator) stress and $\Delta\varepsilon_a$ = recoverable axial strain. In detail,

$$\Delta\sigma_a = \frac{F_{cyclic}}{A} \tag{2}$$

$$\Delta\varepsilon_a = \frac{\delta_{elastic}^{average}}{l_0} \tag{3}$$

where F_{cyclic} = cyclic axial force, A = cross sectional area, $\delta_{elastic}^{average}$ = average elastic (recoverable) axial displacement and l_0 = original gage length. Since axial strain is determined from axial displacement, it is important to measure it accurately.

Displacement readings are usually obtained from linear variable differential transformers or LVDTs [4,5]. For the M_R test, three LVDTs should be placed at equi-angular positions around two parallel aluminum collars, which are attached to the specimen (Fig. 2). On the lower collar, columns are mounted below the LVDTs as contacts for the spring-loaded tips of the sensors. This arrangement allows the two collars to move independently of each other. Spacers maintain a parallel distance (gage length) between the collars while the apparatus is placed on the specimen.

Non-uniformity of Displacement

M_R test data typically display non-uniform displacement histories between three LVDT readings during the loading sequences (Fig. 3). Because the M_R value is calculated from the axial displacement of a specimen during cyclic loading, it is critical to have reliable displacement values from at least three LVDTs (two LVDTs are not sufficient to evaluate the non-uniformity).

Consider the boundary condition imposed by a rigid platen that can rotate (Fig. 1). The distribution of normal stress varies and the resultant is composed of an axial force and a bending moment. Thus, the total displacement can be decomposed into

$$\delta_{(i)} = \delta_{(i)F} + \delta_{(i)M} \tag{4}$$

where

$\delta_{(i)}$ = total displacement of LVDT 'i'

$\delta_{(i)F}$ = displacement of LVDT 'i'due to the axial force

$\delta_{(i)M}$ = displacement of LVDT 'i'due to the bending moment

Displacement due to the axial force (δ_F) will be the same for the three LVDTs. However, displacement due to the bending moment (δ_M) will depend on the angle of rotation of the platen (θ) and the position of the LVDT relative to the axis of rotation (Fig. 4). To describe the rotated plane, consider three LVDTs positioned at equi-angular positions, 120° apart. Because the axis of rotation is assumed to go through the center of the specimen, displacement of each LVDT due to the bending moment will be decided by the position of the LVDT in relation to the axis of rotation. If an LVDT is on the axis of rotation, displacement due to bending moment is zero, and total displacement will be the same as axial displacement. If an LVDT is located on a line perpendicular to the axis of rotation, displacement due to the bending moment will be either maximum δ_{max} or minimum δ_{min} (Fig. 4).

For a cylindrical specimen of radius R, define angles α, β, and χ as the angles between a line from the center of the specimen to each LVDT and the axis of rotation such that the location of δ_{min} is between LVDT1 and LVDT2. Therefore, the displacements of the three LVDTs are

$$\delta_1 = \delta_F - R \sin(\alpha) \sin(\theta) \tag{5}$$

$$\delta_2 = \delta_F - R \sin(\beta) \sin(\theta)$$

$$\delta_3 = \delta_F + R \sin(\chi) \sin(\theta)$$

and the sum is

$$\delta_1 + \delta_2 + \delta_3 = 3\delta_F - R \sin(\theta) (\sin(\alpha)+\sin(\beta)-\sin(\chi)) \tag{6}$$

For equi-angular placement of the three LVDTs, the last term in equation (6) becomes

$$\sin(\alpha) + \sin(\beta) - \sin(\chi) = \sin(\alpha) + \sin(60°-\alpha) - \sin(120°-\alpha) = 0 \tag{7}$$

From equations (6) and (7),

$$\delta_F = (\delta_1 + \delta_2 + \delta_3)/3 = \delta_{average} \tag{8}$$

Consequently, the displacement due to axial force, even if rotation occurs, is simply the mean of the displacement values from the three LVDTs. This means that the angle of rotation does not affect the value of the axial displacement for stiffness calculations. This does not mean that the angle of rotation should not be limited, as the assumption of uniform deformation may be violated as rotation increases.

Angle of Rotation

To estimate the angle of rotation, note that θ is the angle between the normal vectors of the plane before loading (the horizontal plane) and the rotated plane, defined by the (minimum) three LVDT displacement values. Recalling that a plane is described by

$$Ax + By + Cz + D = 0 \tag{9}$$

the angle between the normals of the two planes is [6]

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}} \tag{10}$$

In addition, a plane passing through three points $\mathbf{P}_i\,(x_i, y_i, z_i)$, $\mathbf{P}_j\,(x_j, y_j, z_j)$, $\mathbf{P}_k\,(x_k, y_k, z_k)$ is determined by

$$\begin{vmatrix} y_i & z_i & 1 \\ y_j & z_j & 1 \\ y_k & z_k & 1 \end{vmatrix} x + \begin{vmatrix} z_i & x_i & 1 \\ z_j & x_j & 1 \\ z_k & x_k & 1 \end{vmatrix} y + \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix} z = \begin{vmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{vmatrix} \tag{11}$$

The plane before loading is the horizontal plane:

$$z = 0 \tag{12}$$

The plane at a particular load is defined by the three LVDT readings:

$$LVDT_1 = (R, 0, \delta_1) \tag{13}$$

$$LVDT_2 = \left(-\frac{R}{2}, \frac{R\sqrt{3}}{2}, \delta_2\right) \tag{14}$$

$$LVDT_3 = \left(-\frac{R}{2}, -\frac{R\sqrt{3}}{2}, \delta_3\right) \tag{15}$$

Thus, the equation of the rotated plane at a particular load is

$$\sqrt{3}R\left(\frac{\delta_2}{2} + \frac{\delta_3}{2} - \delta_1\right)x + \frac{3}{2}R(\delta_3 - \delta_2)y + \frac{3\sqrt{3}}{2}R^2 z - \frac{\sqrt{3}}{2}R^2(\delta_1 + \delta_2 + \delta_3) = 0 \tag{16}$$

Substituting equations (12) and (16) into equation (10), the angle of rotation θ is

$$\cos\theta = \frac{\frac{3}{2}R}{\sqrt{\delta_1^2 + \delta_2^2 + \delta_3^2 - \delta_1\delta_2 - \delta_1\delta_3 - \delta_2\delta_3 + \frac{9}{4}R^2}} \tag{17}$$

The axis of rotation is the line of intersection of the rotated plane with the horizontal plane, with

$$z = \frac{\delta_1 + \delta_2 + \delta_3}{3} \tag{18}$$

The equation for the intersection of two planes in the *xy* plane is [6]

$$\begin{vmatrix} C_1 & C_2 \\ A_1 & A_2 \end{vmatrix} x + \begin{vmatrix} C_1 & C_2 \\ B_1 & B_2 \end{vmatrix} y + \begin{vmatrix} C_1 & C_2 \\ D_1 & D_2 \end{vmatrix} = 0 \tag{19}$$

Substituting equations (16) and (18) into equation (19) results in the equation for the axis of rotation:

$$\sqrt{3}R\left(\frac{\delta_2}{2} + \frac{\delta_3}{2} - \delta_1\right)x + \frac{3}{2}R(\delta_3 - \delta_2)y = 0 \tag{20}$$

In summary, from three sensors placed equi-angular to measure axial displacement, the angle of rotation and the position of the axis of rotation can be calculated.

Uniformity Ratio

In NCHRP 1-28A, the uniformity ratio, γ, is given as

$$\gamma = \frac{\delta'_{\max}}{\delta'_{\min}} \tag{21}$$

where $\delta'_{max,\ min}$ are the maximum and minimum displacements *measured* by two LVDTs; $\gamma \leq 1.1$ defines an acceptable test [3]. The uniformity ratio $\gamma_{\max}$ is introduced based on the maximum and minimum displacements *calculated* from three LVDTS. If rotation occurs during the load application, γ values can vary depending on where the LVDTs are located with reference to the axis of rotation. Even if the test result shows that γ is within some limit, the result from the same specimen may not satisfy the condition if $\gamma_{\max}$ is estimated. Thus, it is more reasonable to limit the rotation θ rather than γ. Given a certain amount of allowable rotation (say 0.04°), the uniformity ratio will depend on the amount of recoverable (average) axial displacement (Fig. 5). The same value of rotation could result in different values of $\gamma_{\max}$ depending on the stiffness of the specimen and the deviator stress, both of which influence recoverable axial strain.

Concluding Remarks

In applying load or displacement for an element test, some rotation of the rigid platen may occur such that the fundamental stress field is perturbed by bending. Thus, non-uniformity between displacement measurements may be present, and the readings may consist of a component due to the axial force and a component due to the bending moment. To estimate this non-uniformity during an element test, three LVDTs are needed and for equi-angular placement the sensors, the mean of the three readings is equal to the displacement from the axial stress; the rotation does not affect the mean value. Furthermore, the amount of rotation from bending with respect to the amount of displacement from axial force causes an increase in the uniformity ratio, γ, the value of which is influenced by the location of the LVDTs with reference to the axis of rotation. Therefore, it is more important to limit the degree of rotation and not set a target value for γ.

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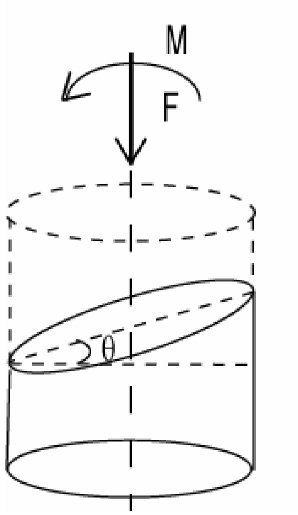

Figure 1. Axial force and bending moment imposed by rigid platens that rotate.

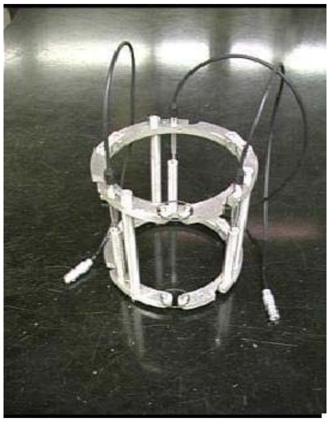

Figure 2. Apparatus for holding LVDTs.

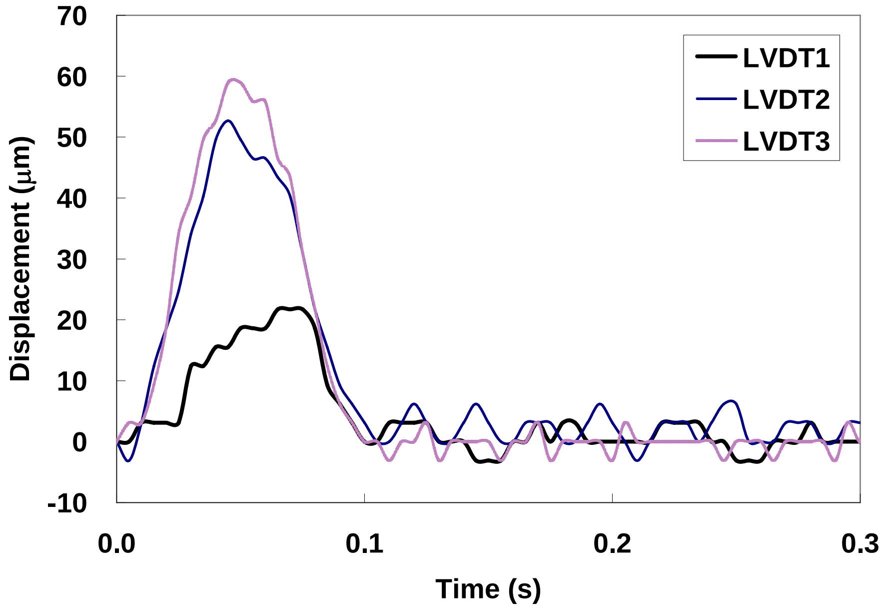

Figure 3. Three LVDT displacement time histories.

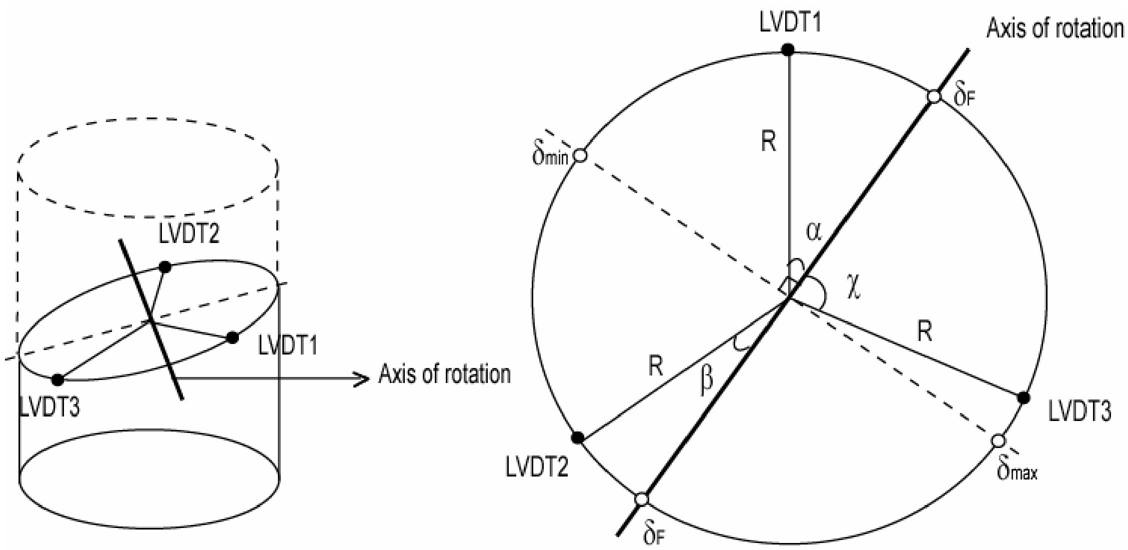

Figure 4. Geometry of specimen and LVDTs with respect to the axis of rotation.

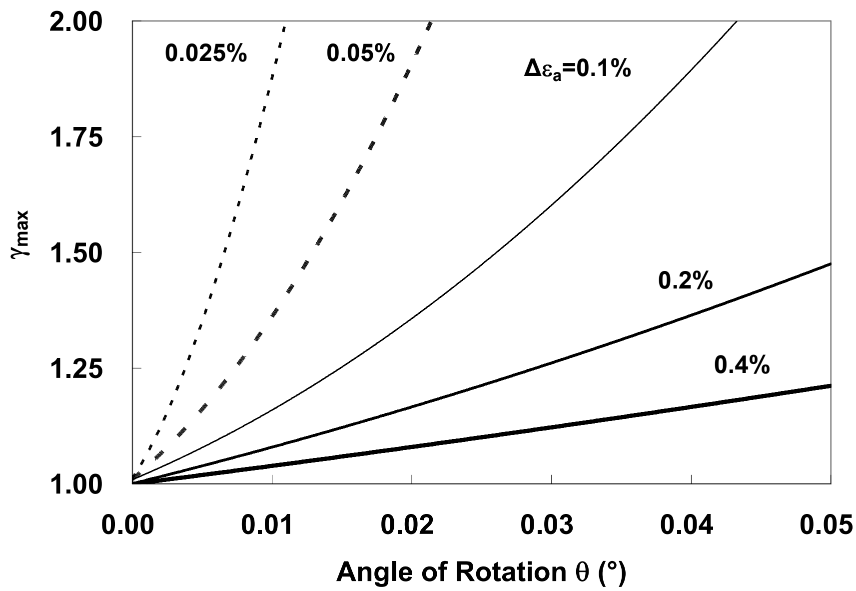

Figure 5. Influence of rotation on the uniformity ratio γ_{max} at various levels of axial strain $\Delta\varepsilon_a$ (gage length = 100 mm).